Reiknirit, rökfræði og reiknanleiki

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skil 7

1 Exercise 5.15 bls 165

Prove that the following three set are not recursive by explicit reduction from the set K - do not use Rice's theorem.

1) $\{x|\phi_x$ is a constant function$\}$

reik frá K til T
$T = \{x|\phi_x$ is a constant function$\}$

$\Theta(x, y) = 1 + \text{Zero}(\text{U}(x, x))$
$= \varphi_i(x, y) = \varphi_{s(i,x)}(y) = \varphi_{s_{s(i)}(x)}(y) = \begin{cases} 1 & x \in k \\ \perp & x \notin k \end{cases}$
$s_{s(i)}(x) \in T <=> x \in K$
Hef sýnt fram á að $s_{s(i)}(x)$ sér red frá K til T.

2) $\{x|\phi_x$ is not the totally undefined function$\}$

reik frá K til T
$T = \{x|\phi_x$ is not the totally undefined function$\}$

$\Theta(x, y) = 1 + \text{Zero}(\text{U}(x, x))$
$= \varphi_i(x, y) = \varphi_{s(i,x)}(y) = \varphi_{s_{s(i)}(x)}(y) = \begin{cases} 1 & x \in k \\ \perp & x \notin k \end{cases}$
$s_{s(i)}(x) \in T <=> x \in K$
Hef sýnt fram á að $s_{s(i)}(x)$ sér red frá K til T.

3) $\{x|$there is y with $\phi_x(y)\downarrow$ and such that ϕ_y is total$\}$

Fyrst undir dæmi er R.E. $T = \{x|\phi_x$ is total$\}$. Þá er allt dæmið það líka

reik frá K til T
$T = \{x|\phi_x$ is total$\}$

$\Theta(x, y) = 1 + \text{Zero}(\text{U}(x, x))$
$= \varphi_i(x, y) = \varphi_{s(i,x)}(y) = \varphi_{s_{s(i)}(x)}(y) = \begin{cases} 1 & x \in k \\ \perp & x \notin k \end{cases}$
$s_{s(i)}(x) \in T <=> x \in K$
Hef sýnt fram á að $s_{s(i)}(x)$ sér red frá K til T.

2 Exercise 5.16 bls 165

For each of the following set and its complement, classify them as recursive, non-recursive but r.e, or non-r.e. You may use Rice's theorem to prove that a set is not recursive. To prove that a set is r.e., show that it is range or domain of partial recursive function. For the rest, use closure results or reductions.

2) $\{x|\phi_x$ is injective$\}$

By Rice's theorem, this set is not recursive: the property is a property of functions, not a property of programs, and is non-trival (there are both injective and

non-injective functions). It follows by closure that the complement is also non-recursive. We need to be a bit more specific about injectivity before we decide whetehr or not the set or its complement is r.e., the defintion of injectivity normally given is $\forall x, y, (x \neq y) \Longrightarrow (f(x) \neq f(y))$, which does not carry over well to non-total functions. We can simply ignore all arguments on which the function is not defined and state that ϕ is injective if, whenever $\phi(x) \downarrow$ and $\phi(y) \downarrow$ then $(x \neq y) \Longrightarrow (\phi(x) \neq \phi(y))$ or we can state that only total functions can be injective. We look at both interpretations below.
Under the first interpretation (only defined values count), we claim that the complement is r.e., so that the set itself cannot be r.e. To see that the complement is r.e., note that placing some programs ϕ_i is this complement set can be done just by indentifying two arguments x and y such that $x \neq y$ and yet $\phi_i(x) = \phi_i(y)$. We can do this by quintuple dovetailing, on all programs, all pairs of arguments, and all steps for each agrument; whenever we find a z such that

step$(\prod_1^5(z), \prod_2^5(z), \prod_3^5(z)) \neq 0$ and
step$(\prod_1^5(z), \prod_4^5(z), \prod_5^5(z)) \neq 0$ and
$\prod_2^5(z) \neq \prod_4^5(z)$ and
step$(\prod_1^5(z), \prod_2^5(z), \prod_3^5(z)) =$ step$(\prod_1^5(z), \prod_4^5(z), \prod_5^5(z))$,
we print $\prod_1^5(z)$

Under the secound iterpretation, however, neither the set nor its complement ir r.e. Thus we must do some extra work. We reduce $\bar{K}$ in turn to the set and to its complement. To reduce $\bar{K}$ to the set of (total) injective functions, we define.
$\Theta(x, y) = \begin{cases} y & \text{step}(x, x, y) = 0 \\ \phi_j(y) & \text{otherwise} \end{cases}$
where ϕ_j is totally undefined function. This new functions is clearly partial recursive (a valid defintion by cases), so we can write $\Theta(x, y) = \phi_i(x, y)$ for some index i. Using the s-m-n construction, we now get $\phi_{s(i,x)}(y) = \Theta(xmy)$. But note that $\phi_{s(i,x)}$ is the identity function whenever $x \in \bar{K}$ and is undefined on at least one value of y otherwise. Hence $s(i, x)$ is injective whenever $x \in \bar{K}$ is not injective (because it is partial) otherwise. Thus we have reduced $\bar{K}$ to our set of injective functions, thereby showing that our set is not r.e. It remaining to show that the complement of our set is not r.e. either; for that we need to reduce $\bar{K}$ directly to the complements. We define a new function $\zeta(x, y) = y + \text{Zero}(\phi_{\text{univ}}(x, x))$. Again, this is a partial recursive function, so that there exists an index k such that $\phi_{k(x,y)} = \zeta(x, y)$; again, the s-m-n theorem tells us that we can write $\phi_{s(k,x)}(y) = \phi_k(x, y) = \zeta(x, y)$. But note that $\phi_{s(k,x)}(y)$ is totally undefined when $x \in \bar{K}$ and is the identity functions otherwise; that is, $s(k, x)$ is in the complement of our set whenever $x \in \bar{K}$ so that we have a reduction from $\bar{K}$ to the complement of our set, thereby proving it to be non-r.e.

6) The set of all r.e. sets that contain at least three elements.

Recall that an r.e. set is the range (or domain) of a partial recursive function. Thus our set could equally we have been defined as the set of all partial recursive functions that have at least three elements in their range (or domain). Sinve having a range (or domain) of size at least three is and I/O property, this set is not recurisve by Rice's theorem. The set is r.e., however: intuitively, we can dovetail on all functions, all arguments, and all numbers of steps, keeping track of what converged and printing out a function index as soon as it has converged on at least three distinct arguments (for domain definitions, but range defintions are similar). Formalizing the notion is not hard (we basically want to run step$(\prod_1^3(z), \prod_2^3(z), \prod_3^3(z))$,

recording $\prod_1^3(z)$ and one of $\prod_2^3(z)$ (for domains) or step($\prod_1^3(z), \prod_2^3(z), \prod_3^3(z)$) (for ranges) for which the function is nonzero, and printing $\prod_1^3(z)$ as soon as trhee distinct values of $\prod_2^3(z)$ (for domains) or of step($\prod_1^3(z), \prod_2^3(z), \prod_3^3(z)$) (for ranges) are found that pair with the same ($\prod_1^3(z)$ value.
Since the set is non-recursive and r.e., its complement it not r.e. This is an r.e. set. Informally, since we can enumerate all Turing machines and enumerate all possible input strings, we can start a triple dovetailing process, on all machines, all inputs, and all steps. In the process, we every now and then discover a pair (machine, input) such that the machine accepts the string; we then "chalk one up" for that machine. As soon as we have chalked up three for a machine, we print it (its index). In that way, we correctly enumerate all machines that accept at least three strings.

7) The set of all partial recursive functions with finite domain.

Neither this set nor its complement is r.e. That neither is recursive is an immediate consequence of Rice's theorem, since being defined on a finite of infinite number of inputs is an I/O behavior. To show that neither S nor $\bar{S}$ is r.e., we reduce $\bar{K}$ to each in turn.
To reduce $\bar{K}$ to S, we define the function $\Theta(x, y) = \phi_{\text{univ}}(x, x)$; since this is a valid partial recursive function, it has some index ϕ_i and we can write $\phi_{s(x)}(y) = \phi_i(x, y) = \Theta(x, y)$. Now, if $x \in \bar{K}$, then $\phi_{s(x)}(y)$ diverges for all y, and thus has empty (and hence finite) domain, and thus $s(x)$ belongs to S. On the other hand, if $x \in K$, then $\phi_{s(x)}(y) = c$, where $c = \phi_x(x)$ is a constant, so that $\phi_{s(x)}$ is total and thus has infinite domain, and hence $s(x) \notin S$, as desired.
To reduce $\bar{K}$ to $\bar{S}$, we define the function.

$$\Theta(x, y) = \begin{cases} \text{Zero}(y) & \text{step}(x, x, y) = 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

since this is a valid partial recursive function, is has some index ϕ_y and we can write $\phi_{s(x)}(y) = \phi_i(x, y) = \Theta(x, y)$. Now if $x \in \bar{K}$ then $\phi_{s(x)}(y) = \text{Zero}(y)$ and thus is total and has infinite domain, and thus $s(x)$ belongs to $\bar{S}$. On the other hand, if $x \in \bar{K}$ then $\phi_{s(x)}(y)$, is undefined for all $y \geq y0$ for some contstant $y0$, and thus has finite domain and therefore $s(x)$ does not belongs to $\bar{S}$, as desired.